

Section One: Calculator – free

(50 marks)

This section has **six (6)** questions. Answer all questions. Write your answers in the spaces provided.

Working time: 50 minutes

Question 1

(7 marks)

(a) Simplify $\frac{3}{x^2-4} - \frac{5}{x+2}$ (3)

$$= \frac{3 - 5(x-2)}{(x+2)(x-2)} \checkmark$$
$$= \frac{13-5x}{x^2-4} \checkmark$$

(b) Simplify $\frac{3m^2-6m-24}{m^2-5m+4} \div \frac{m^2-m-6}{m^2-3m}$ (4)

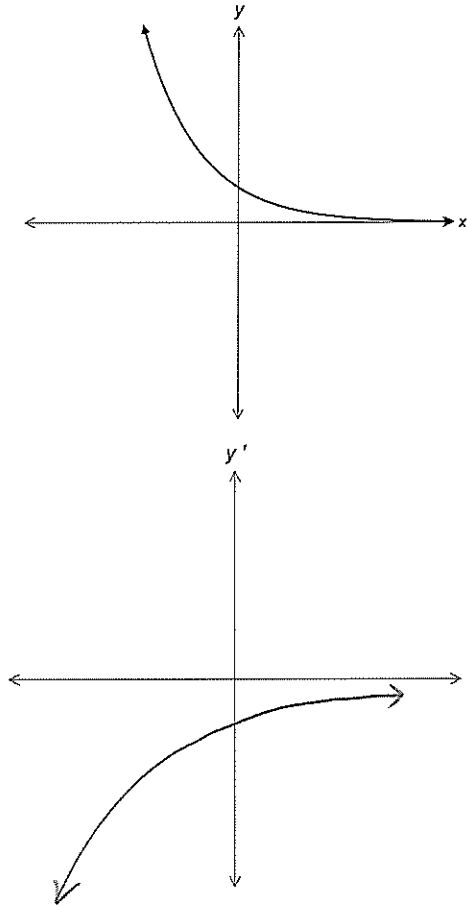
$$\frac{3(m-4)(m+2)}{(m/4)(m-1)} \times \frac{m(m-3)}{(m/3)(m+2)} \checkmark$$
$$= \frac{3m}{m-1} \checkmark$$

$$\begin{aligned} m &\neq 4 \\ m &\neq -2 \\ m &\neq 3 \end{aligned} \checkmark$$

Question 2

(9 marks)

- (a) Sketch the graph of the derivative function (for the function shown) on the axes provided. (2)



shape ✓
all below x-axis ✓

- (b) Differentiate the following with respect to x .

(i) $f(x) = \frac{-x}{x^2+1}$ (express in simplest form) (3)

$$f'(x) = \frac{-1(x^2+1) - 2x(-x)}{(x^2+1)^2} \checkmark \checkmark \quad -1 \text{ per error}$$

$$= \frac{-x^2 - 1 + 2x^2}{(x^2+1)^2}$$

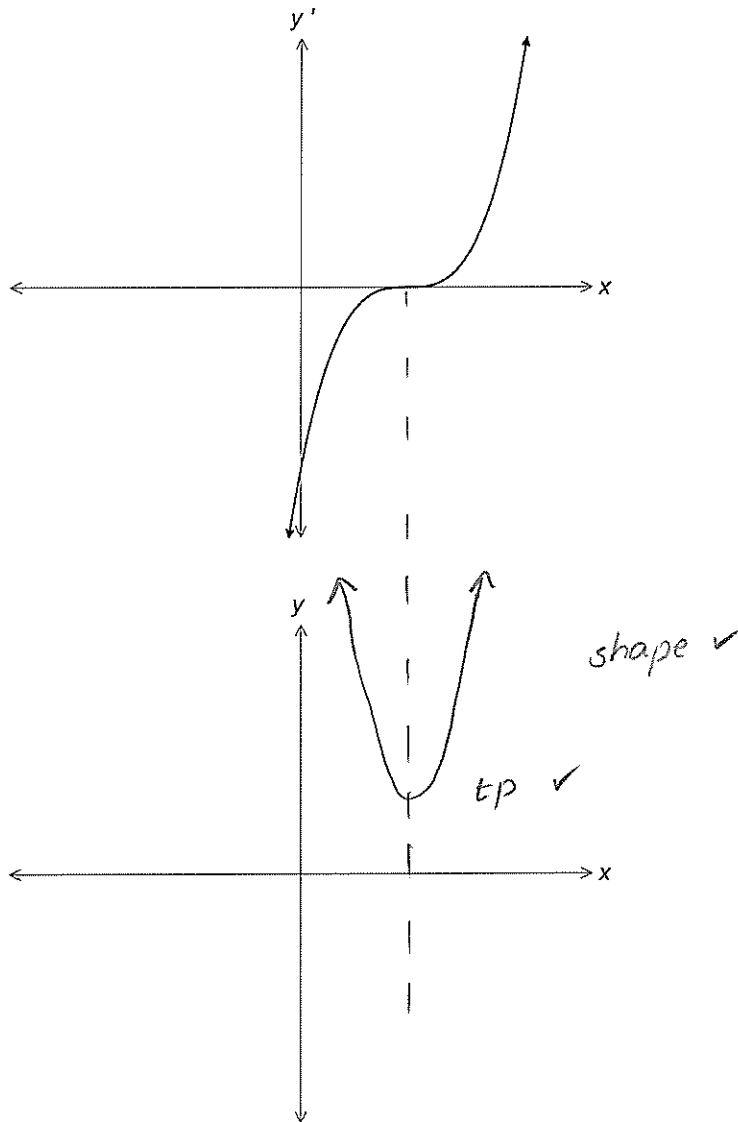
$$= \frac{x^2 - 1}{(x^2+1)^2} \checkmark$$

(ii) $g(x) = (x+1)^2 e^{x^2}$ (do not simplify) (2)

$$g'(x) = 2(x+1)e^{x^2} + 2x e^{x^2} (x+1)^2 \checkmark \checkmark$$

Question 2 (continued)

- (c) Given the derivative function, sketch the graph of the function on the axes provided. (2)



Question 3

(12 marks)

Consider the curve $y = x^3 - 2x^2 - 4x + 3$

- (a) It is claimed that the tangent to the curve at the point where $x = 1$ passes through the point $(3, 8)$. Is this claim valid? Justify your answer. (4)

$$y' = 3x^2 - 4x - 4 \quad \checkmark$$

$$\begin{array}{l} \text{at } x = 1 \quad y = -2 \quad \checkmark \\ \text{at } x = 1 \quad y' = -5 \quad \checkmark \end{array}$$

$$y = -5x + c$$

subst $(1, -2)$

$$-2 = -5(1) + c$$

$$c = 3$$

$$\therefore y = -5x + 3 \quad \checkmark$$

subst $x = 3$

$$y = -15 + 3 = -12 \neq 8$$

\therefore claim not valid \checkmark

- (b) Determine the value of x for which y is a maximum. (4)

$$\text{max when } 3x^2 - 4x - 4 = 0$$

$$(3x + 2)(x - 2) = 0$$

$$x = -\frac{2}{3} \text{ or } x = 2 \quad \checkmark$$

$$y'' = 6x - 4 \quad \checkmark$$

$$\text{if } x = -\frac{2}{3} \quad y'' < 0 \quad \therefore \text{max}$$

$$\text{if } x = 2 \quad y'' > 0 \quad \therefore \text{min}$$

\checkmark tests both points

$$\therefore \text{max when } x = -\frac{2}{3} \quad \checkmark$$

Question 3 (continued)

(c) Solve the following system of equations.

(4)

$$x - 2y + 4z = 2 \quad \textcircled{1}$$

$$2x + y + 3z = -1 \quad \textcircled{2}$$

$$-x - y - 2z = 0 \quad \textcircled{3}$$

$$2 \times \textcircled{1} - \textcircled{2} \quad -5y + 5z = 5 \quad \textcircled{4}$$
$$\div 5 \Rightarrow -y + z = 1 \quad \textcircled{5} \quad \checkmark$$

$$\textcircled{1} + \textcircled{3} \quad -3y + 2z = 2 \quad \textcircled{6}$$
$$2 \times \textcircled{5} \quad -2y + 2z = 2 \quad \textcircled{7} \quad \checkmark$$

$$\textcircled{6} - \textcircled{7} \quad -y = 0$$
$$\therefore y = 0 \quad \checkmark$$

subst $y = 0$ into $\textcircled{5}$

$$z = 1 \quad \checkmark$$

subst $y = 0, z = 1$ into $\textcircled{3}$

$$-x - 2 = 0$$
$$x = -2 \quad \checkmark$$

Question 4

(10 marks)

- (a) Determine c given that the graph of $f(x) = cx^2 + x^{-2}$ has a point of inflection at $(1, f(1))$. (3)

$$f'(x) = 2cx - 2x^{-3} \quad \checkmark$$

$$f''(x) = 2c + 6x^{-4} \quad \checkmark$$

$$\text{Let } 2c + \frac{6}{x^4} = 0$$

$$\text{when } x = 1 \quad 2c + 6 = 0$$
$$c = -3 \quad \checkmark$$

Question 4 (continued)

(b) The functions $f(x)$ and $g(x)$ are defined as follows

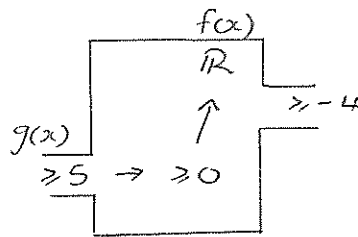
$$f(x) = x^2 - 4 \text{ and } g(x) = \sqrt{x-5}$$

(i) Determine the simplified expressions for $f[g(x)]$ and $g[f(x)]$. (3)

$$\begin{aligned} f(g(x)) &= (\sqrt{x-5})^2 - 4 \checkmark \\ &= x - 5 - 4 \\ &= x - 9 \checkmark \end{aligned}$$

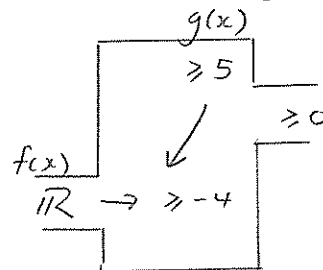
$$\begin{aligned} g(f(x)) &= \sqrt{x^2 - 4 - 5} \\ &= \sqrt{x^2 - 9} \checkmark \end{aligned}$$

(ii) Determine the range of $f[g(x)]$. (2)



Range $y: y \geq -4, y \in \mathbb{R}$
✓✓

(iii) Determine the domain of $g[f(x)]$. (2)



$$\begin{aligned} x^2 - 4 &\geq 5 \\ x^2 &\geq 9 \end{aligned}$$

domain
 $x: x \geq 3 \text{ or } x \leq -3 \quad x \in \mathbb{R}$
✓ ✓

Question 5

-1 no +c

(7 marks)

(a) Determine $\int (1+3x^2)(x-2) dx$ (3)

$$\begin{aligned} &= \int (3x^3 - 6x^2 + x - 2) dx \checkmark \\ &= \frac{3x^4}{4} - 2x^3 + \frac{x^2}{2} - 2x + c \checkmark \checkmark \end{aligned}$$

(b) Determine $\int 4x^3(3x^4 - 5)^7 dx$ (2)

$$\begin{aligned} &= \frac{1}{3} \int 12x^3(3x^4 - 5)^7 dx \checkmark & f(x) &= 3x^4 - 5 \\ & & f'(x) &= 12x^3 \\ &= \frac{\frac{1}{3} (3x^4 - 5)^8}{8} + c \\ &= \frac{(3x^4 - 5)^8}{24} + c \checkmark \end{aligned}$$

(c) Determine $\int 12xe^{x^2} dx$ (2)

$$\begin{aligned} &= 6 \int 2x e^{x^2} dx \checkmark & f(x) &= x^2 \\ &= 6 e^{x^2} + c \checkmark & f'(x) &= 2x \end{aligned}$$

Question 6

(5 marks)

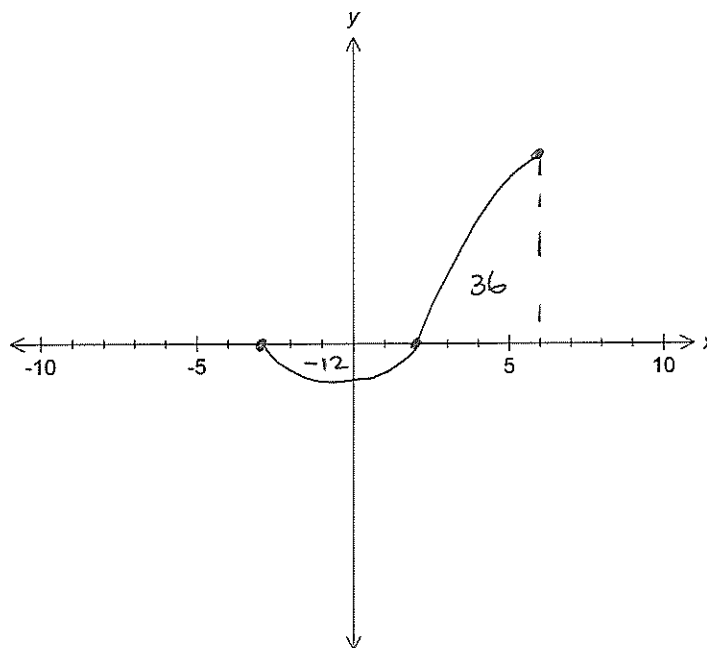
$f(x)$ is defined such that $\int_{-3}^6 f(x) dx = 24$ and $\int_2^6 f(x) dx = 36$

(a) Find

$$\begin{aligned} \text{(i)} \quad \int_{-3}^2 f(x) dx &= 24 - 36 \\ &= -12 \quad \checkmark \end{aligned} \tag{1}$$

$$\begin{aligned} \text{(ii)} \quad \int_{-3}^2 (4f(x)+3) dx & \tag{3} \\ &= \int_{-3}^2 4f(x) dx + \int_{-3}^2 (3) dx \\ &= 4(-12) + [3x]_{-3}^2 \quad \checkmark \quad \checkmark \\ &= -48 + (6 - -9) \\ &= -33 \quad \checkmark \end{aligned}$$

(b) Sketch a possible graph of $y=f(x)$ for $-3 \leq x \leq 6$. Your graph should display the relative areas of important regions but you do not need to draw this graph to scale. (1)



Section Two: Calculator – assumed

(100 marks)

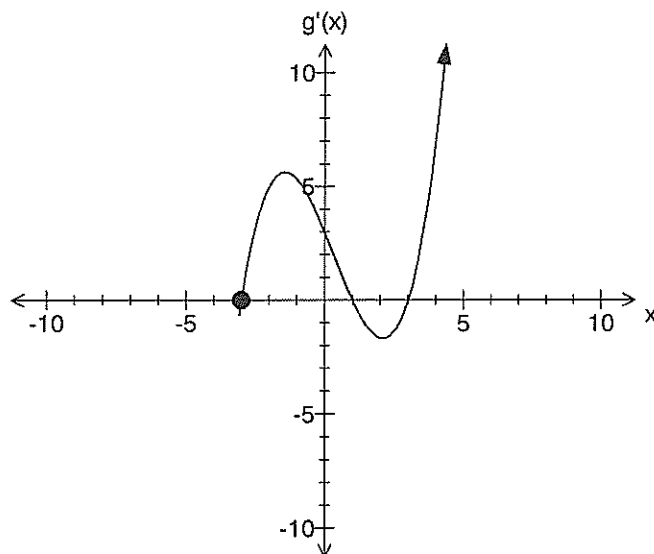
This section has **ten (10)** questions. Answer all questions. Write your answers in the spaces provided.

Working time: 100 minutes

Question 7

(7 marks)

The graph of $g'(x)$ is given below.



- (a) What can be said about the gradient of the function $g(x)$ between $x = -3$ to $x = 1$? (1)

positive ✓

- (b) When does the function, $g(x)$ have a negative gradient? (2)

$1 < x < 3$ ✓✓ (\leq)

- (c) State an equation for the tangent to the graph of $g(x)$ at $x = 3$. (1)

$y = k$, k a constant ✓

- (d) Find the value of x at which $g(x)$ has a relative maximum for $-3 \leq x \leq 4$ (1)

$x = 1$ ✓

- (e) Find the x -coordinate of each point of inflection of the graph of $g(x)$ for $-3 \leq x \leq 4$ (2)

*$x = -1.5$ ✓
 $x = 2$ ✓*

Question 8

(12 marks)

- (a) Events A and B are such $P(A) = \frac{1}{2}$, $P(B) = \frac{7}{12}$ and $P(\overline{A \cup B}) = \frac{1}{4}$

Show that events A and B are not mutually exclusive.

(3)

$$P(A \cup B) = \frac{3}{4}$$

$$P(A \cap B) = P(A) + P(B) - P(A \cup B)$$

$$= \frac{1}{2} + \frac{7}{12} - \frac{3}{4} \checkmark$$

$$= \frac{1}{3} \checkmark$$

$\neq 0 \therefore$ not mutually exclusive \checkmark

- (b) A toy robot has 3 main components (X , Y and Z) which are manufactured separately and then assembled together. Previous random testing of components has shown that:

$$P(X \text{ defective}) = 0.002, \quad P(Y \text{ defective}) = 0.015, \quad P(Z \text{ defective}) = 0.003$$

If a toy robot is selected at random, what is the probability that:

- (i) only component Y is defective,

(2)

$$0.998 \times 0.015 \times 0.997 \checkmark$$
$$= 0.01492509$$

- (ii) at least one of its components are defective.

(2)

$$1 - \text{none defective}$$
$$= 1 - 0.998 \times 0.985 \times 0.997 \checkmark$$
$$= 0.0199 \text{ (4dp)} \checkmark$$

Question 8 (continued)

(c) If X and Y are independent events and $P(X) = 0.75$ and $P(X \cup Y) = 0.875$, find

(i) $P(Y)$ (3)

if independent then

$$P(X) \cdot P(Y) = P(X \cap Y)$$

$$0.75 P(Y) = P(X \cap Y) \quad \checkmark$$

$$P(X \cup Y) = P(X) + P(Y) - P(X \cap Y)$$

$$0.875 = 0.75 + P(Y) - 0.75 P(Y) \quad \checkmark$$

$$0.125 = 0.25 P(Y)$$

$$P(Y) = 0.5 \quad \checkmark$$

(ii) $P(Y|X) = 0.5 \quad \checkmark$ (1)

(iii) $P(X|Y') = 0.75 \quad \checkmark$ (1)

Question 9

(12 marks)

- (a) It takes 12 hours to drain a storage tank by opening the valve at the bottom. The depth, y , of fluid in the tank t hours after the valve is opened is given by

$$y = 6\left(1 - \frac{t}{12}\right)^2 \text{ metres.}$$

- (i) Find the rate $\frac{dy}{dt}$ m/hour at which the tank is draining at time, t . (2)

$$\begin{aligned} \frac{dy}{dt} &= 12 \left(1 - \frac{t}{12}\right)' \cdot -\frac{1}{12} \quad \checkmark \\ &= -\left(1 - \frac{t}{12}\right) \quad \checkmark \\ &= \frac{t}{12} - 1 \end{aligned}$$

- (ii) When is the fluid in the tank falling fastest and slowest?

What are the values of $\frac{dy}{dt}$ at these times? (3)

$$0 \leq t \leq 12$$

$$\frac{t}{12} - 1 = 0$$

$$t = 12 \quad \checkmark$$

minimum, slowest
 $\frac{dy}{dt} = 0 \quad \checkmark$

fastest

$$\text{at } t = 0$$

$$\frac{dy}{dt} = -1 \quad \checkmark$$

Question 9 (continued)

- (c) If $y = kx^3$ for some constant k , use the incremental formula to establish the percentage increase in x required to yield a 15% increase in y . (3)

$$\begin{aligned}\delta y &= \frac{dy}{dx} \delta x \\ 0.15y &= 3kx^2 \delta x \quad \checkmark \\ 0.15kx^3 &= 3kx^2 \delta x \quad \checkmark \\ \delta x &= 0.05x \\ &5\% \text{ change in } x \text{ required} \quad \checkmark\end{aligned}$$

- (d) A company sells goods such that its revenue, in dollars, from selling x items is given by the equation,

$$\begin{aligned}R(x) &= 5x(20x - x^2) \\ &= 100x^2 - 5x^3\end{aligned}$$

- (i) Determine the marginal revenue when $x = 10$. (2)

$$\begin{aligned}R'(x) &= 200x - 15x^2 \quad \checkmark \\ R'(10) &= 500 \\ \therefore \text{marginal revenue is } \$500 \quad \checkmark\end{aligned}$$

- (ii) What does marginal revenue represent? (2)

The revenue for the 11th item
is approximately \$500 ✓

Question 10

(7 marks)

The Australian Kayak team must select 4 elite rowers from 14 possible contenders to be the new 'Awesome Foursome'.

- (a) How many different selections are possible? (1)

$$\binom{14}{4} \checkmark = 1001$$

Mike is the singles kayak champion and Geoff is the runner up champion.

- (b) What is the probability that of the 4 rowers chosen at random:

- (i) Mike is included? (1)

$$\frac{\binom{1}{1}\binom{13}{3}}{\binom{14}{4}} \checkmark = \frac{286}{1001} = 0.2857$$

- (ii) Mike and Geoff are included? (1)

$$\frac{\binom{2}{2}\binom{12}{2}}{\binom{14}{4}} \checkmark = \frac{66}{1001} = 0.0659$$

- (iii) Mike or Geoff is selected? (2)

$$\frac{286}{1001} + \frac{286}{1001} - \frac{66}{1001} \checkmark \checkmark = \frac{506}{1001} = 0.5055$$

OR

$$1 - \frac{\binom{2}{0}\binom{12}{4}}{\binom{14}{4}} = 0.5055$$

- (c) If Mike is selected for the Kayak team, what is the probability that Geoff is also selected? (2)

$$\frac{\frac{66}{1001}}{\frac{278}{1001}} \checkmark = \frac{66}{286} \checkmark = 0.2307.$$

Question 11

(12 marks)

(a) The function $f(x)$ is differentiable for all $x \in \mathbb{R}$ and satisfies the conditions

$$f'(x) < 0 \text{ where } x < 2$$

$$f'(x) = 0 \text{ where } x = 2$$

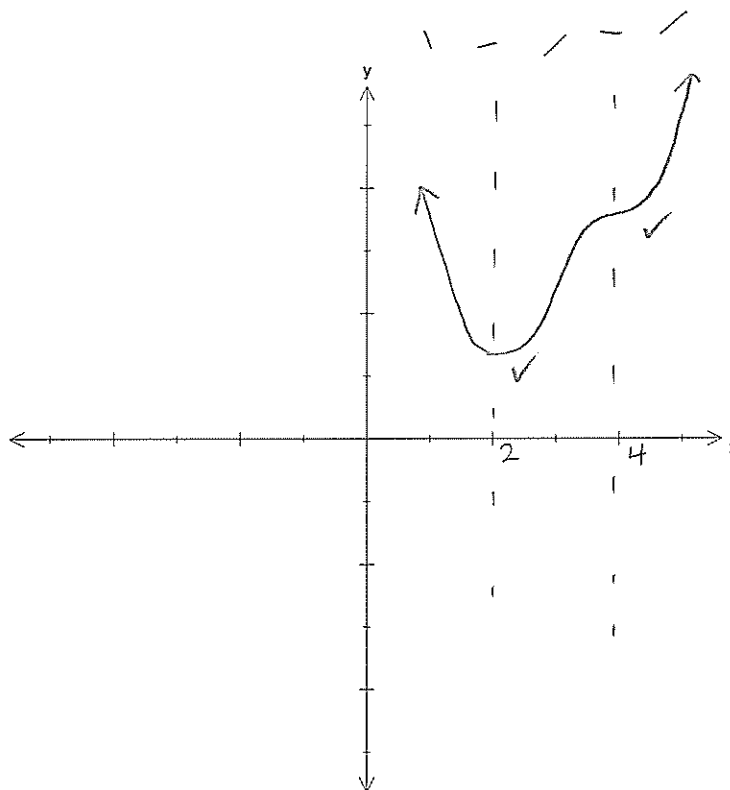
$$f'(x) = 0 \text{ where } x = 4$$

$$f'(x) > 0 \text{ where } 2 < x < 4$$

$$f'(x) > 0 \text{ where } x > 4$$

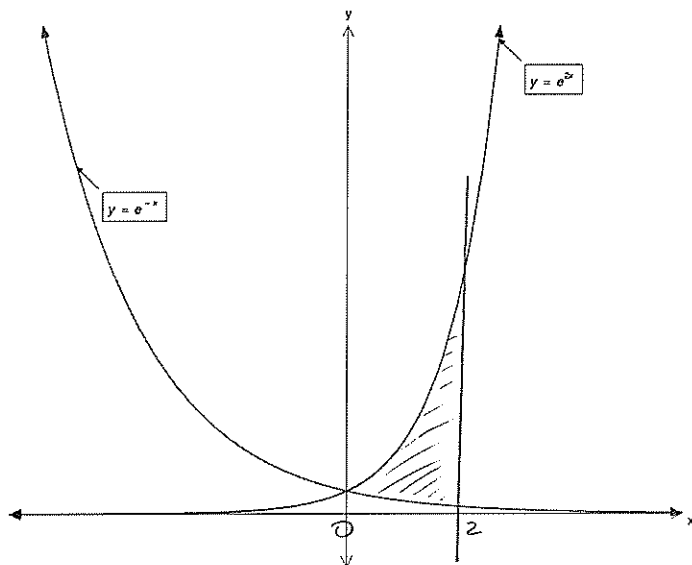
(i) Draw a sketch of this function $f(x)$.

(3)



Question 11 (continued)

(b) The curve $y = e^{2x}$ and $y = e^{-x}$ intersect at the point $(0, 1)$ as shown in the diagram.



Find the area enclosed by the curves and the line $x=2$.
Leave your answer in terms of e .

(4)

$$\begin{aligned} & \int_0^2 (e^{2x} - e^{-x}) dx \quad \checkmark \\ & = \left[\frac{e^{2x}}{2} - (-e^{-x}) \right]_0^2 \quad \checkmark \\ & = \frac{e^4}{2} + e^{-2} - \left(\frac{1}{2} + 1 \right) \\ & = \frac{e^4}{2} + \frac{1}{e^2} - \frac{3}{2} \quad \checkmark \end{aligned}$$

Question 11 (continued)

- (c) State the resulting equation when the graph of $y = e^x$ undergoes the following transformations **in succession**: (3)

- horizontal dilation of factor $\frac{1}{3}$
- reflection about the y-axis
- vertical translation 5 units in the direction of the negative y-axis
- horizontal translation 3 units in the direction of the positive x-axis
- vertical dilation of factor 2

$$y = e^{3x} \quad \checkmark$$

$$y = e^{-3x} \quad \checkmark$$

$$y = e^{-3x} - 5 \quad \checkmark$$

$$y = e^{-3(x-3)} - 5 \quad \checkmark$$

$$y = 2(e^{-3(x-3)} - 5) \quad \checkmark$$

$$y = 2e^{-3(x-3)} \div 10$$

- (d) The point $(3, 0.5e^4)$ lies on the curve of $y = 0.5e^{x+1}$. Identify the subsequent location of this point if the transformations listed below are applied **in succession**. (2)

- reflection about the x-axis
- horizontal translation 7 units in the direction of the negative x-axis
- vertical translation 3 units in the direction of the positive y-axis
- reflection about the y-axis

$$(3, -0.5e^4)$$

$$(-4, -0.5e^4)$$

$$(-4, -0.5e^4 + 3)$$

$$(-4, -0.5e^4 + 3)$$

✓ ✓

Question 12

(9 marks)

Research has been conducted to determine the benefits of a flu vaccine before winter for adults over 65. The following information has been obtained:

60% of the target population (i.e. adults over 65) had the flu vaccine and of these 22% actually developed the flu, 3% developed a chest infection and the remainder had no flu-like symptoms over the winter.

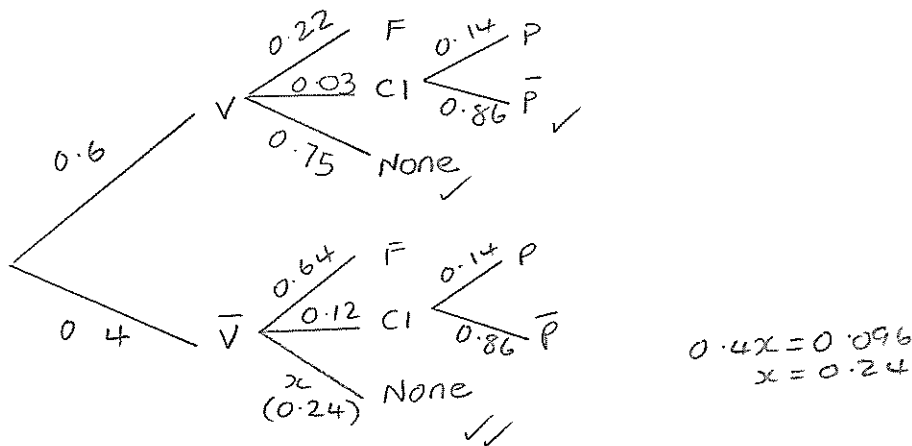
Of those who did not have the flu vaccine 12% developed a chest infection.

The proportion of those studied who did not have the vaccine and had no flu-like symptoms over the winter was 0.096.

14% of all those who developed a chest infection also got pneumonia.

(Note that in this same sample no one developed both the flu and a chest infection)

- (a) Draw a tree diagram to represent the above information. (4)



- (b) For a randomly chosen person from this study determine the probability that:

- (i) the person developed the flu if they did not have the flu vaccine. (1)

0.64 ✓

- (ii) the person had the flu vaccine and developed pneumonia. (1)

$0.6 \times 0.03 \times 0.14 = 0.00252$ ✓

- (iii) the person had the vaccine if they developed pneumonia. (3)

$$\frac{0.00252 \checkmark}{0.00252 + 0.4 \times 0.12 \times 0.14} = 0.2727$$

Question 13

(12 marks)

Consider the letters of the word POLICE.

How many arrangements are there of these 6 letters (without repetition) if

- (a) each arrangement must end with a vowel? (1)

$$\frac{5}{5!} \times \frac{4}{4!} \times \frac{3}{3!} \times \frac{2}{2!} \times \frac{1}{1!} \times \frac{3}{3!} \checkmark = 360$$

- (b) the vowels in each arrangement must be together? (2)

O I E , P , L , C

$$4! \times 3! = 144$$

- (c) the vowels must be separated by the consonants (ie 2 vowels must not be together)? (2)

V C V C V C

$$3! \times 3! \times 2 = 72$$

Now suppose that 4 letters are chosen from this word and that the **order of selection is unimportant**.

- (d) How many different 4 letter groups are possible if

- (i) there is no restriction? (1)

$$\binom{6}{4} = 15$$

- (ii) there must be at least one vowel? (2)

$$\binom{3}{1} \binom{3}{3} + \binom{3}{2} \binom{3}{2} + \binom{3}{3} \binom{3}{1} \checkmark$$

$$= 15$$

- (e) Determine the probability that in the four letter selection that is made, whenever O appears, E also appears. (4)

$$\frac{\binom{2}{2} \binom{4}{2} + \binom{2}{0} \binom{4}{4} + \binom{1}{0} \binom{1}{1} \binom{4}{3}}{\binom{6}{4}} \checkmark \checkmark \checkmark$$

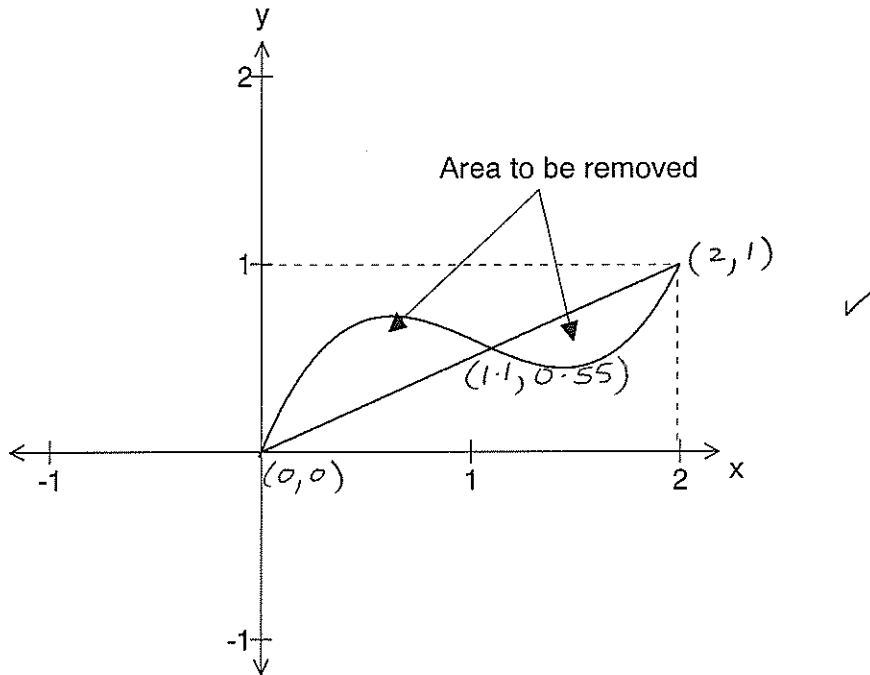
$$= \frac{11}{15} \checkmark$$

Question 14

(11 marks)

- (a) A dressmaker wishes to cut a section of cloth from a piece of material measuring 2 metres by one metre. The curved edges of the piece of cloth to be removed are defined as being between the following equations:

$$y_1 = 0.5x \text{ and } y_2 = x^3 - 3.1x^2 + 2.7x$$



- (i) Label the 3 points of intersection with co-ordinates. (1)
- (ii) Write an integral that would give the area of region bound by the two functions. (2)

$$A = \int_0^{1.1} (x^3 - 3.1x^2 + 2.7x - 0.5x) dx + \int_{1.1}^2 (0.5x - (x^3 - 3.1x^2 + 2.7x)) dx$$

✓ -1 per error

- (iii) Calculate the area of the cloth removed, correct to 2 decimal places. (2)

$$A = 0.51 \text{ m}^2 \quad \checkmark \quad -\frac{1}{2} \text{ units}$$

Question 14 (continued)

(b) A group of anthropologists found that human tooth size is continuing to decrease, such that $\frac{dS}{dt} = kS$. In Northern Europeans, for example, tooth size reduction now has a rate of 1% per 1000 years.

(i) If t represents time in years and S represents tooth size, find the value of k , rounded to 8 decimal places. (2)

$$S = S_0 e^{kt}$$
$$0.99S_0 = S_0 e^{1000k} \quad \checkmark$$
$$k = -0.00001005 \quad \checkmark$$

(ii) In how many years will human tooth size be 90% of their present size? (2)

$$0.9 = e^{-0.00001005t} \quad \checkmark$$
$$t = 10483.3 \text{ years} \quad \checkmark$$

($t = 10483.6$ using (i))

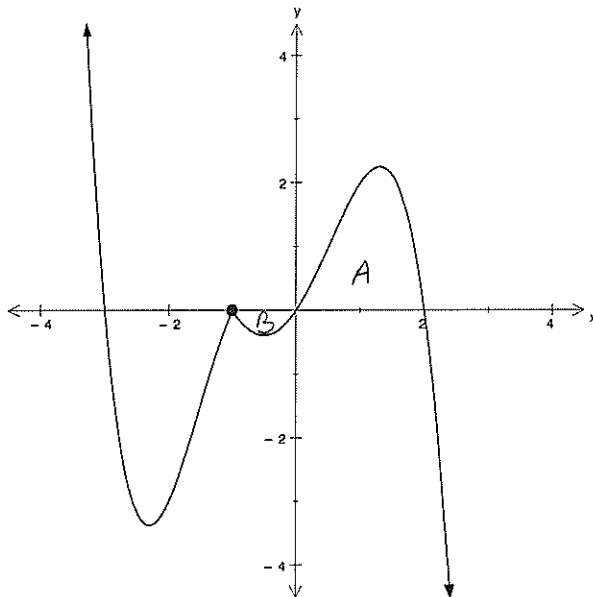
(iii) What will be our descendant's tooth size 20 000 years from now? (as a percentage of our present tooth size) (2)

$$S = S_0 e^{-0.00001005 \times 20000} \quad \checkmark$$
$$= 0.8179 \quad \checkmark$$
$$= 81.79\% \quad \checkmark$$

Question 15

(8 marks)

(a) For the function $y = f(x)$ below



It is known that

$$\int_{-3}^{-1} f(x) dx = -75$$

$$\int_{-1}^2 f(x) dx = 20$$

The area between the curve and the x -axis from $x = -1$ to $x = 2$ is 80 square units.

Use the information above and mathematical reasoning to determine the value of each of the following.

(i) $\int_{-1}^0 f(x) dx$ $B - A = 20$ ✓ (3)
 $= -30$ $A + B = 80$ ✓
 $A = 30$ ✓
 $B = 50$ ✓

(ii) the area between the curve and the x -axis from $x = -3$ to $x = 0$ (1)

$$75 + 30 = 105 \quad \checkmark$$

Question 15 (continued)

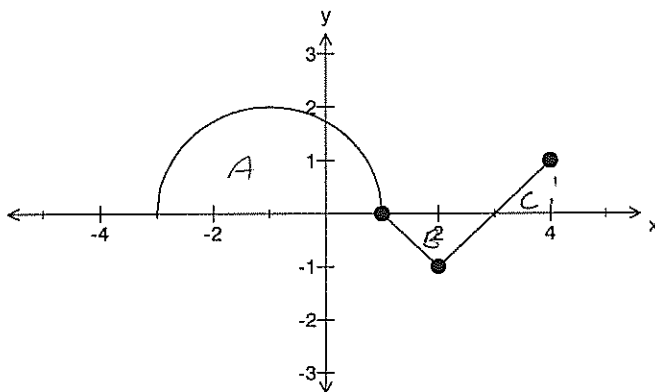
(iii) $\int_{-3}^2 f(x) dx$ (1)

$$= -75 - 30 + 50$$

$$= -55 \quad \checkmark$$

- (b) The graph of a function $f(x)$ consists of a semi-circle and two line segments as shown.

Find the exact value of $\int_{-3}^4 f(x) dx$ (3)



$$A = \frac{1}{2} \times \pi \times 2^2 = 2\pi$$

$$B = \frac{1}{2} \times 2 \times 1 = 1$$

$$C = \frac{1}{2} \times 1 \times 1 = \frac{1}{2} \quad \checkmark$$

$$\int_{-3}^4 f(x) dx = A + (-B) + C$$

$$= 2\pi - 1 + \frac{1}{2}$$

$$= 2\pi - \frac{1}{2} \quad \checkmark$$

Question 16


(10 marks)

A piece of wire 8cm long is cut into two unequal parts. One part is used to form a rectangle that has a length three times its width. The other part of the wire is used to form a square.

- (a) If the width of the rectangle is x units, show that the equation that will give the sum of the areas of the rectangle and the square in terms of x is: (5)

$$A = 7x^2 - 8x + 4$$

$$x \begin{array}{|c|} \hline 3x \\ \hline \square \\ \hline 3x \\ \hline \end{array} x = 8x \quad \therefore \text{Square} = 8 - 8x$$

$$\frac{8-8x}{4} = 2-2x$$


$$\begin{aligned}
 A &= 3x(x) + (2-2x)^2 \\
 &= 3x^2 + 4 - 8x + 4x^2 \\
 &= 7x^2 - 8x + 4 \text{ as required}
 \end{aligned}$$

- (b) Using Calculus, find the length of each part of the wire when the sum of the areas is a minimum. (5)

$$\frac{dA}{dx} = 14x - 8$$

$$14x - 8 = 0$$

$$x = \frac{4}{7}$$

$$\frac{d^2A}{dx^2} = 14 > 0 \therefore \text{min}$$

length of wire for rectangle $\frac{32}{7}$ cm
 square $\frac{24}{7}$ cm